Connectedness of fractals associated with Arnoux-Rauzy substitutions

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January 11, 2011

Abstract

Rauzy fractals are compact sets with fractal boundary that can be associated with any unimodular Pisot irreducible substitution. These fractals can be defined as the Hausdorff limit of a sequence of compact planar sets, where each set is the projection of a finite union of faces of unit cubes. We exploit this combinatorial definition to prove the connectedness of the fractal associated with any finite product of Arnoux-Rauzy substitutions.

1 Introduction

Rauzy fractals are compact sets with fractal boundary that can be associated with any unimodular Pisot irreducible substitution. (See Definition 2.7 and 2.9 below for precise definitions.) They first appeared in the work of Rauzy [Rau82], who generalized the theory of interval exchange transformations by defining a *domain* exchange transformation of three pieces in $\mathbb{R}^2$. Each of the three pieces is translated along a vector (one distinct vector for each piece) in order to give a different partition of the same shape (see Figure 1). These fractals were also discussed in the later work of Thurston [Thu89] in the context of numeration systems in non-integer bases.

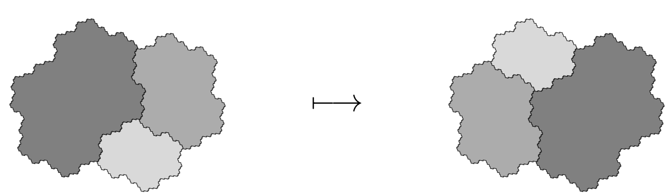

Figure 1: Domain exchange in the Tribonacci fractal.

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Properties of Rauzy fractals Originally, Rauzy considered the *Tribonacci substitution* $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$, and he proved in [Rau82] that the dynamics of the symbolic dynamical system generated by σ is realized by the domain exchange in the Rauzy fractal associated with σ. Rauzy's results have then been generalized by Arnoux and Ito [AI01] as follows: for any unimodular Pisot irreducible substitution σ on d letters, the subshift X_σ generated by σ is measure-theoretically isomorphic to a domain exchange of d pieces in the fractal associated with σ, on the assumption that σ verifies a combinatorial condition called the *strong coincidence condition* (see also [CS01]).

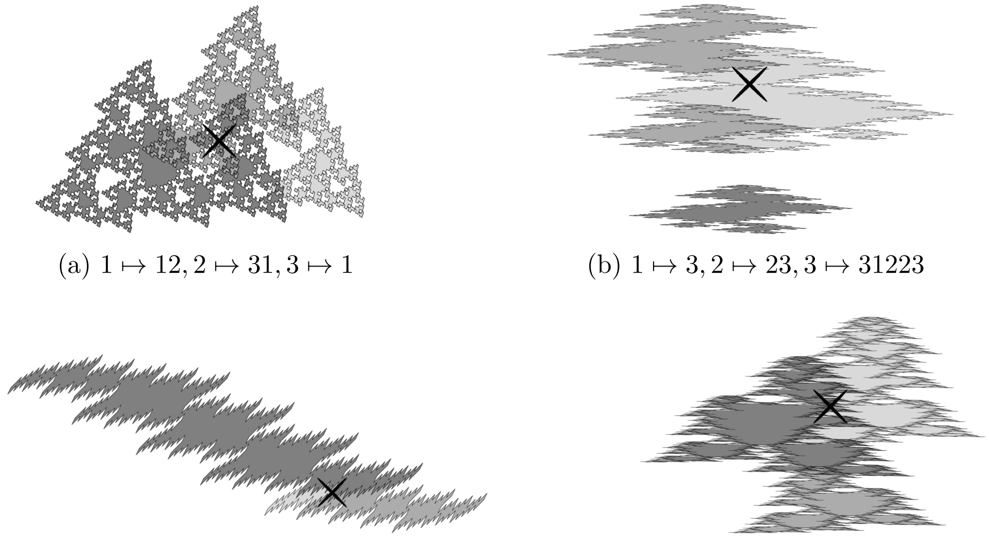
Rauzy fractals enjoy other nice dynamical properties. The subshift X_σ is measure-theoretically isomorphic to a translation on the $(d-1)$-dimensional torus if σ satisfies the *super coincidence condition* introduced in [IR06, BK06]. Also, Rauzy fractals provide explicit Markov partitions for some hyperbolic automorphisms of the torus [IO93, Pra99]. It is currently not known if a Pisot irreducible substitution σ always verifies the strong and the super coincidence conditions; the Pisot conjecture states that this is always the case.

Let us mention some other properties of Rauzy fractals. In numeration systems, they provide natural extensions of β-transformations with relevant algebraic properties [ABBS08]. In theoretical physics, they are good candidates for explicit cut-and-project schemes which model quasi-cristals [GVG04]. In discrete geometry, they are related to discrete plane generation via multidimensional continued fraction algorithms [IO93, ABFJ07, Fer09].

Topology of Rauzy fractals The Rauzy fractal associated with the Tribonacci substitution (Figure 1) has a very nice topology: the origin is an inner point, and it is homeomorphic to a closed disc [Mes00]. However, it has been shown that the topology of Rauzy fractals can be very complicated in general. For example, they can fail to be connected or simply connected, and the origin is not always an inner point of the set; see Figure 2. More examples are given in [ST10].

The study of topological properties of Rauzy fractals has many applications. Connectedness of the Rauzy fractal is linked to some properties of the Markov partitions of hyperbolic automorphisms of the torus [Adl98]. Cut-points of the fractal are related to some topological invariants of tiling spaces [BDS09]. In Diophantine approximation, explicit computation of the size of the largest ball contained in the fractal provides the best possible simultaneous approximations of some two-dimensional vectors with respect to a specific norm [HM06]. In number theory, finite greedy expansions in non-integer bases (β-numeration systems) are closely related to the inner points of the fractal, and the connectedness of the fractal is conjectured to guarantee explicit relations between the norm of β and the β-expansion of 1 [AG05]. The properties of rational numbers with purely periodic β-expansions are closely related to the shape of the boundary of the Rauzy fractal [ABBS08, AFSS10]. In discrete geometry, studying the position of the origin in the fractal allows us to study the structure of discrete planes and to generate them [IO93, BLPP10].

Arnoux-Rauzy substitutions Sturmian sequences are a classical object of symbolic dynamics. They are the infinite sequences of two letters with complexity $n+1$ (*i.e.*, they have exactly $n+1$ factors of length n), and they correspond to natural codings of irrational rotations on the circle [MH40]. These sequences are also closely related to continued fraction



(a) $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$

(b) $1 \mapsto 3, 2 \mapsto 23, 3 \mapsto 31223$

(c) $1 \mapsto 21111, 2 \mapsto 31111, 3 \mapsto 1$

(d) $1 \mapsto 123, 2 \mapsto 1, 3 \mapsto 31$

Figure 2: Examples of Rauzy fractals: (a) is connected and has uncountable fundamental group; (b) is disconnected and $\mathbf{0}$ is not an inner point; (c) is connected, has uncountable fundamental group, and $\mathbf{0}$ is an inner point; (d) is connected and $\mathbf{0}$ is not an inner point. Black crosses mark the origin.

expansions of real numbers; see [PF02, Chap. 6] and [Lot97, Chap. 2] for a detailed survey of their properties.

Arnoux-Rauzy sequences were introduced in [AR91] to generalize Sturmian sequences to three-letter alphabets. They are the infinite sequences of three symbols obtained by iterating the *Arnoux-Rauzy substitutions* σ_1, σ_2, σ_3 defined by

$$\sigma_i \ : \ \begin{cases} j & \mapsto \quad j \qquad \text{if} \ \ j = i \\ j & \mapsto \quad ji \qquad \text{if} \ \ j \neq i \end{cases} \qquad (i = 1, 2, 3),$$

with each of the σ_i occuring infinitely often in the iteration. These sequences have complexity $2n+1$ and they yield an algorithm for simultaneous approximations of some particular pairs of algebraic numbers [CFM08]. It was conjectured that Arnoux-Rauzy sequences correspond to natural codings of translations on the two-dimensional torus (as in the Sturmian case with rotations on the circle), but this conjecture has been disproved in [CFZ00]. For more references about Arnoux-Rauzy sequences, see [BFZ05, CC06].

Our results In the case where σ is a two-letter substitution, topological properties of Rauzy fractals (which are subsets of $\mathbb{R}$) are fully understood: the fractal is connected (*i.e.*, an interval) if and only if the substitution is Sturmian [EI98], [PF02, Chap. 9]. However, very few general results are known when σ acts on three or more letters, because topological studies such as the ones cited above usually rely on a description of the boundary of a single fractal, which makes the study of some *families* of fractals very difficult.

In this article, we study the family of Rauzy fractals associated with finite products of Arnoux-Rauzy substitutions and we prove that they are connected (Theorem 4.3).

To this end, we use a combinatorial characterization of Rauzy fractals given by Arnoux and Ito in [AI01] (see Definition 2.9 below). For any unimodular Pisot irreducible substitution σ on d letters, they define a *generalized substitution* $\mathbf{E}_1^*(\sigma)$, that acts not on words but on faces of unit cubes in $\mathbb{R}^d$. The Rauzy fractal associated with σ can then be obtained by iterating $\mathbf{E}_1^*(\sigma)$ starting from a small set of unit faces (for example  when $d = 3$). This gives an increasing sequence of finite sets of unit faces in $\mathbb{R}^d$ which, if projected on a particular hyperplane and renormalized appropriately at each step, admits a Hausdorff limit that is equal to the Rauzy fractal.

Our results are based on an alternative description of $\mathbf{E}_1^*(\sigma)$ substitutions, introduced in [IO93, IO94, ABI02]. It consists of trying to compute the image $\mathbf{E}_1^*(\sigma)(P)$ of a set of unit faces P by *concatenating* the images of the elements of P, instead of using the definition of $\mathbf{E}_1^*(\sigma)$ to compute the new position of each face. This is similar to the relation $\sigma(uv) = \sigma(u)\sigma(v)$ which is valid for words but difficult to generalize to higher dimensions.

Let us note that a proof of Theorem 4.3 has been announced in [Can03], relying on completely different methods, but it has not been published.

Outline of the paper In Section 2, we give the definition of generalized substitutions and explain how they act on discrete planes, we give a definition of the Rauzy fractal using these objects, and we introduce Arnoux-Rauzy substitutions. In Section 3, we establish a combinatorial sufficient condition for the connectedness of Rauzy fractals. In Section 4 we apply the results of Section 3 to prove the connectedness of the fractal associated with an Arnoux-Rauzy substitution (Theorem 4.2 and 4.3). Finally, in Section 5 we provide examples to show that some possible generalizations of Theorem 4.3 are not true.

2 Preliminaries

The definitions and results of this section are valid in any dimension but we state them for dimension 3 only, since our main results (Theorem 4.2 and 4.3) hold in dimension 3.

2.1 Discrete planes and unit faces

We start by giving a geometric definition of discrete planes, following [Rev91, IO93, IO94, ABI02]. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denote the canonical basis of $\mathbb{R}^3$, and recall that the plane of (non-zero) normal vector $\mathbf{v} \in \mathbb{R}_+^3$ is the set of points $\mathbf{x} \in \mathbb{R}^3$ such that $\langle \mathbf{x}, \mathbf{v} \rangle = 0$.

Definition 2.1 (Discrete plane)**.** Let $\mathbf{v} \in (\mathbb{R}_+ \setminus \{0\})^3$ be a non-zero vector, and let $\mathcal{S}$ be the half-space of points $\mathbf{x} \in \mathbb{R}^3$ such that $\langle \mathbf{x}, \mathbf{v} \rangle < 0$. We define the *discrete plane* $\mathcal{P}_\mathbf{v}$ *of normal vector* $\mathbf{v}$ as the boundary of the union of the closed unit cubes with integer coordinates that intersect $\mathcal{S}$.

A discrete plane can be seen as a union of unit faces of three different types. Let $i \in \{1,2,3\}$ and $\mathbf{x} \in \mathbb{Z}^3$. The *unit face type i at point* $\mathbf{x}$ is the set $[\mathbf{x}, i]^*$ defined by

$$\begin{array}{rcl} [\mathbf{x},1]^* & = & \{\mathbf{x} + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 : \lambda, \mu \in [0,1]\} \\ [\mathbf{x},2]^* & = & \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_3 : \lambda, \mu \in [0,1]\} \\ [\mathbf{x},3]^* & = & \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2 : \lambda, \mu \in [0,1]\} \end{array}$$

(see Figure 3). The type i of face $[\mathbf{x}, i]^*$ corresponds to the canonical vector $\mathbf{e}_i$ to which it is orthogonal to. If $\mathbf{x} \in \mathbb{R}^3$, we can write $\mathbf{y} + [\mathbf{x}, i]^*$ instead of $[\mathbf{x} + \mathbf{y}, i]^*$, and we denote by $\mathbf{x} + X$ the translation of a union of faces X by $\mathbf{x}$. Let us remark that Definition 2.1 implies that the set $[\mathbf{0}, 1]^* \cup [\mathbf{0}, 2]^* \cup [\mathbf{0}, 3]^*$ is included in every discrete plane.

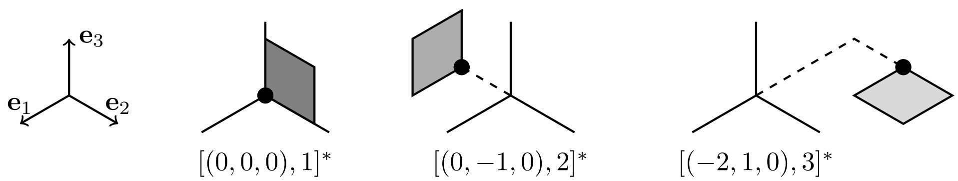


Figure 3: Three unit faces of different types.

The following proposition gives an alternative definition of discrete planes, where the belonging of each face to a plane is translated into an inequality on scalar products.

Proposition 2.2 ([ABI02, ABS04])**.** *The discrete plane $\mathcal{P}_{\mathbf{v}}$ is the union of faces $[\mathbf{x}, i]^*$ satisfying $0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle$.*

2.2 Generalized substitutions

We start with the classical notion of unidimensional substitution. Let $\mathcal{A} = \{1, \ldots, d\}$ be a finite alphabet. A *substitution* is a function $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ such that $\sigma(uv) = \sigma(u)\sigma(v)$ for every words $u, v \in \mathcal{A}^*$, and such that the image of each letter of $\mathcal{A}$ is non-empty. The *incidence matrix* $\mathbf{M}_\sigma$ of σ is the square matrix of size $d \times d$ defined by $\mathbf{M}_\sigma = (m_{ij})$, where m_{ij} is the number of occurrences of the letter i in $\sigma(j)$. A substitution is said *unimodular* if the determinant of its incidence matrix equals 1 or -1.

A classical example of a substitution is the Tribonacci substitution introduced by Rauzy in [Rau82], whose action on $\{1,2,3\}^*$ and incidence matrix are given by

$$\sigma : \left\{ \begin{array}{lcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{array} \right. \qquad \text{and} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We now introduce *generalized substitutions*, which act not on unidimensional words, but on unit faces in $\mathbb{R}^3$. This formalism was sketched by Ito and Ohtsuki [IO93, IO94], and then refined later by Arnoux and Ito in [AI01] (see Definition 2.3 below), where they also highlight the connections between generalized substitutions and discrete planes (Propositions 2.5 and 2.6 below).

Definition 2.3 (Generalized substitution)**.** Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ be a unimodular substitution. The *generalized substitution* associated with σ, denoted by $\mathbf{E}_1^*(\sigma)$, is defined by

$$\mathbf{E}_1^*(\sigma)([\mathbf{x},i]^*) \;=\; \bigcup_{j=1,2,3} \; \bigcup_{s|\sigma(j)=pis} [\mathbf{M}_\sigma^{-1}(\mathbf{x}+\ell(s)),j]^*.$$

We extend this definition to any union of unit faces:

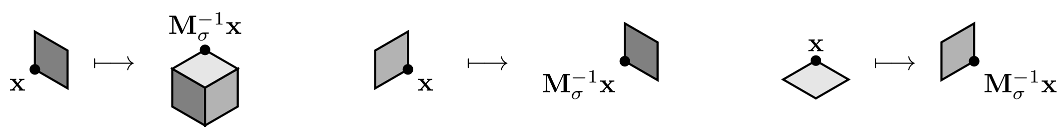
$$\mathbf{E}_1^*(\sigma)(X_1 \cup X_2) \;=\; \mathbf{E}_1^*(\sigma)(X_1) \cup \mathbf{E}_1^*(\sigma)(X_2).$$

Let us remark that $\mathbf{E}_1^*(\sigma)$ is completely described by $\mathbf{M}_\sigma$ and the images of the faces $[\mathbf{0},1]^*$, $[\mathbf{0},2]^*$ and $[\mathbf{0},3]^*$, because $\mathbf{E}_1^*(\sigma)([\mathbf{x},i]^*) = \mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{E}_1^*([\mathbf{0},i]^*)$ for every unimodular substitution σ and every face $[\mathbf{x},i]^*$. It is also worth noticing that $\mathbf{E}_1^*(\sigma\circ\sigma') = \mathbf{E}_1^*(\sigma')\circ\mathbf{E}_1^*(\sigma)$ holds for every unimodular substitutions σ and σ'; see [AI01] for more details.

Example 2.4. Let σ be the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. The action of $\mathbf{E}_1^*(\sigma)$ on unit faces is given by

$$\begin{array}{rcl} \mathbf{E}_1^*(\sigma)([\mathbf{x},1]^*) & = & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1,0,-1),1]^* \cup [(0,1,-1),2]^* \cup [(0,0,0),3]^* \\ \mathbf{E}_1^*(\sigma)([\mathbf{x},2]^*) & = & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),1]^* \\ \mathbf{E}_1^*(\sigma)([\mathbf{x},3]^*) & = & \mathbf{M}_\sigma^{-1}\mathbf{x} + [(0,0,0),2]^* \end{array},$$

which can be represented graphically as follows:



In general, the images of two distinct faces are not necessarily disjoint, but it is the case when the faces belong to a common discrete plane, as stated in Proposition 2.5.

Proposition 2.5 ([AI01])**.** *If $[\mathbf{x},i]^*$ and $[\mathbf{x}',i']^*$ are two distinct faces of $\mathcal{P}_\mathbf{v}$, then the sets $\mathbf{E}_1^*(\sigma)([\mathbf{x},i]^*)$ and $\mathbf{E}_1^*(\sigma)([\mathbf{x}',i']^*)$ are disjoint up to a set of measure zero.*

Proposition 2.6 states that the image of a discrete plane by a generalized substitution is again a discrete plane.

Proposition 2.6 ([AI01])**.** *Let $\mathcal{P}_\mathbf{v}$ be a discrete plane and σ be a unimodular substitution. We have*

$$E_1^*(\sigma)(\mathcal{P}_\mathbf{v}) = \mathcal{P}_{{}^tMv}.$$

2.3 The Rauzy fractal associated with a substitution

We will now give the definition of the Rauzy fractal associated with a unimodular Pisot irreducible substitution, as in [AI01]. We recall that a *Pisot number* is a real algebraic integer greater than 1 whose conjugates have absolute value less than 1.

Definition 2.7 (Pisot irreducible substitution)**.** A unimodular substitution σ is *Pisot irreducible* if the maximal eigenvalue of $\mathbf{M}_\sigma$ is a Pisot number, and if the characteristic polynomial of $\mathbf{M}_\sigma$ is irreducible.

Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ be a unimodular Pisot irreducible substitution, and let β be the maximal eigenvalue of $\mathbf{M}_\sigma$. We denote by $\mathbf{u}_\beta$ an eigenvector of $\mathbf{M}_\sigma$ associated with β, and by $\mathbf{v}_\beta$ an eigenvector of ${}^{\mathrm{t}}\mathbf{M}_\sigma$ associated with β.

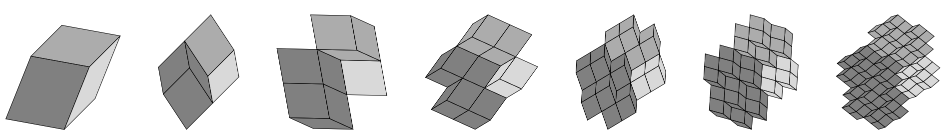
Definition 2.8 (Contracting plane)**.** Let σ be a unimodular Pisot irreducible substitution. The *contracting plane* $\mathbb{P}_\mathrm{c}$ associated with σ is the plane of normal vector $\mathbf{v}_\beta$. We denote by $\pi_\mathrm{c} : \mathbb{R}^3 \to \mathbb{P}_\mathrm{c}$ the projection of $\mathbb{R}^3$ on $\mathbb{P}_\mathrm{c}$ along $\mathbf{u}_\beta$.

Let $\mathcal{U} = [\mathbf{0},1]^* \cup [\mathbf{0},2]^* \cup [\mathbf{0},3]^*$. Proposition 2.6 enables us to iterate $\mathbf{E}_1^*(\sigma)$ on $\mathcal{U}$ in order to obtain an infinite sequence of patterns of increasing size that are included in the discrete plane of normal vector $\mathbf{v}_\beta$, *i.e.*, the discretization of the contracting plane. Let us remark that the set $\mathcal{P}_{\mathbf{v}_\beta}$ is indeed a discrete plane, because $\mathbf{v}_\beta$ has positive coordinates. This can be proved by applying the Perron-Frobenius theorem to $\mathbf{M}_\sigma$, because the matrix of a unimodular Pisot irreducible substitutions is always primitive; see [CS01].

It is possible to project and renormalize the patterns by applying $\mathbf{M}_\sigma \circ \pi_\mathrm{c}$, in order to obtain a sequence of subsets of the contracting plane $\mathbb{P}_\mathrm{c}$ that converges to a compact subset of $\mathbb{P}_\mathrm{c}$. More precisely, for $n \geqslant 0$, let

$$\mathcal{D}_n \;=\; \mathbf{M}_\sigma^n \circ \pi_\mathrm{c} \circ \mathbf{E}_1^*(\sigma)^n(\mathcal{U}).$$

Arnoux and Ito proved in [AI01] that $(\mathcal{D}_n)_{n\geqslant 0}$ is a convergent sequence in the metric space of compact subsets of $\mathbb{P}_\mathrm{c}$ together with the Hausdorff distance, as illustrated below.



Definition 2.9 (Rauzy fractal)**.** Let $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ be a unimodular Pisot irreducible substitution. The *Rauzy fractal* associated with σ is the Hausdorff limit of the sequence $(\mathcal{D}_n)_{n\geqslant 0}$.

2.4 Arnoux-Rauzy substitutions

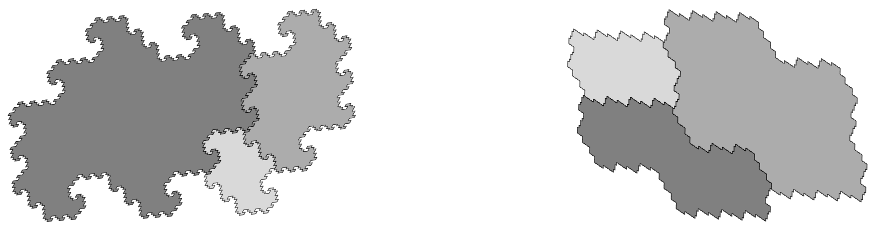
Let σ_1, σ_2, σ_3 be the *Arnoux-Rauzy substitutions* [AR91] defined by

$$\sigma_1 : \left\{\begin{array}{lcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{array}\right., \qquad \sigma_2 : \left\{\begin{array}{lcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{array}\right., \qquad \sigma_3 : \left\{\begin{array}{lcl} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{array}\right..$$

The following proposition enables us to define Rauzy fractals associated with finite products of Arnoux-Rauzy substitutions.

Proposition 2.10 ([AI01])**.** *A finite product of Arnoux-Rauzy substitutions where each σ_i appears at least once is a unimodular Pisot irreducible substitution.*

As we will see in the following sections, the Rauzy fractal associated with such a product of Arnoux-Rauzy substitutions is always connected (Theorem 4.3), but not necessarily simply connected (Section 5). Examples of these fractals are depicted in Figure 1, Figure 2c and Figure 4.



(a) $\sigma_1\sigma_1\sigma_2\sigma_2\sigma_3\sigma_3$ (b) $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_2\sigma_1\sigma_3$

Figure 4: Examples of Rauzy fractals associated with products of Arnoux-Rauzy substitutions.

3 Covering by a set of patterns

A *pattern* is a finite set of unit faces. In the following, we will not make the distinction between a pattern and the union of its elements.

The aim of this section is to introduce the notion of $\mathcal{L}$-*covering* of a pattern by a set of patterns $\mathcal{L}$ (Definition 3.1), and the notion of *stability* of a set of patterns with respect to a generalized substitution (Definition 3.3). We then use these concepts to give a simple sufficient condition for the connectedness of a pattern (Proposition 3.5), which will be used in Section 4. The notion of $\mathcal{L}$-covering already appeared in [IO93, IO94, ABI02, ABS04], but for patterns consisting of two faces only.

Definition 3.1 ($\mathcal{L}$-covering)**.** Let D be a union of unit faces and $\mathcal{L}$ be a set of patterns. An $\mathcal{L}$-*chain* from a face $e \in D$ to a face $f \in D$ is a finite sequence of patterns $(p_1, \ldots, p_n) \in \mathcal{L}^n$ such that:

1. $e \in p_1$ and $f \in p_n$;
2. p_k and p_{k+1} share at least one face, for all $k \in \{1, \ldots, n-1\}$;
3. $p_k \in D$ for all $k \in \{1, \ldots, n\}$.

We say that D is $\mathcal{L}$-*covered* if for all faces $e, f \in D$, there exists an $\mathcal{L}$-chain from e to f.

Roughly speaking, D being $\mathcal{L}$-covered means that we can connect any two faces of D by a “path” made of patterns of $\mathcal{L}$ in which two consecutive patterns share at least one face.

The following lemma states that concatenations of $\mathcal{L}$-chains remain $\mathcal{L}$-chains.

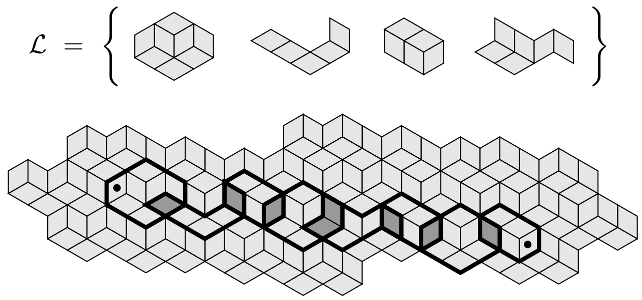


Figure 5: Example of an $\mathcal{L}$-chain. Dark grey indicates the intersection of two patterns.

Lemma 3.2 ([IO94])**.** *Let $\mathcal{L}$ be a set of patterns, D be a union of unit faces, and e, f, g three faces of D. If there exists an $\mathcal{L}$-chain from e to f and an $\mathcal{L}$-chain from f to g, then there exists an $\mathcal{L}$-chain from e to g.*

Proof. Let $(p_1, \dots, p_n)$ be an $\mathcal{L}$-chain from e to f, and $(q_1, \dots, q_m)$ an $\mathcal{L}$-chain from f to g. An $\mathcal{L}$-chain from e to g is given by $(p_1, \dots, p_n, q_1, \dots, q_m)$, because f is in $p_n \cap q_1$.

Let Σ be a generalized substitution and D be an $\mathcal{L}$-covered pattern. We want to know when $\Sigma(D)$ is $\mathcal{L}$-covered. It turns out that there is a simple and easy to verify sufficient condition to check this, namely the *stability* of $\mathcal{L}$ under Σ (Definition 3.3), as stated in Proposition 3.4 below.

Definition 3.3 (Stability)**.** Let Σ be a generalized substitution. A set of patterns $\mathcal{L}$ is *stable* under Σ if $\Sigma(p)$ is $\mathcal{L}$-covered for all $p \in \mathcal{L}$.

Proposition 3.4. *Let $\mathcal{L}$ be a set of patterns that is stable under a generalized substitution Σ, and let D be an $\mathcal{L}$-covered union of unit faces. Then $\Sigma(D)$ is $\mathcal{L}$-covered.*

Proof. Let f and f' be two faces of $\Sigma(D)$. To prove that $\Sigma(D)$ is $\mathcal{L}$-covered, we need to construct an $\mathcal{L}$-chain from f to f'. Let e and e' be two faces of D such that $f \in \Sigma(e)$ and $f' \in \Sigma(e')$. Since D is $\mathcal{L}$-covered, there exists an $\mathcal{L}$-chain $(p_1, \dots, p_n)$ from e to e'. For all $k \in \{2, \dots, n-1\}$, let f_k be a face of $\Sigma(p_k \cap p_{k+1})$, and let $f_1 = f$, $f_n = f'$. Such a face f_k exists because $(p_1, \dots, p_n)$ is an $\mathcal{L}$-chain.

For all $k \in \{1, \dots, n-1\}$, there exists an $\mathcal{L}$-chain from f_k to f_{k+1}, because f_k and f_{k+1} are in $\Sigma(p_{k+1})$ and $\mathcal{L}$ is stable under Σ (see Figure 6). Lemma 3.2 implies that the concatenation of the $\mathcal{L}$-chains from f_k to f_{k+1} is an $\mathcal{L}$-chain from f to f'.

The stability of a given set $\mathcal{L}$ under Σ is easy to verify: it can be done in finite time, since there are only a finite number of patterns to check, and a pattern can be covered only in finitely many ways by patterns of $\mathcal{L}$.

The following Proposition relates $\mathcal{L}$-coverings and connectedness: if a pattern D is $\mathcal{L}$-covered by a set of patterns $\mathcal{L}$ whose elements have a connected projection by π_c, then $\pi_c(D)$ is also connected.

Proposition 3.5. *Let $\mathcal{L}$ be a set of patterns and D be an $\mathcal{L}$-covered union of unit faces. If $\pi_c(p)$ is connected for all $p \in \mathcal{L}$, then $\pi_c(D)$ is connected.*

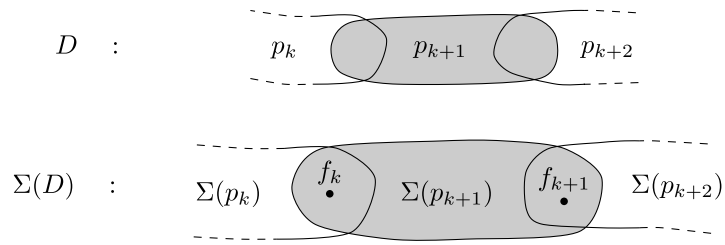


Figure 6: Illustration of the proof of Proposition 3.4.

Proof. Let x, y be two points of $\pi_c(D)$ and e, f two faces of D such that $x \in \pi_c(e)$ and $y \in \pi_c(f)$. Let $(p_1, \ldots, p_n)$ be an $\mathcal{L}$-chain from e to f. The sets $\pi_c(p_i)$ are connected and $\pi_c(p_i) \cup \pi_c(p_{i+1}) \neq \varnothing$ for all i, so there exists an arc from $x \in \pi_c(p_1)$ to $y \in \pi_c(p_n)$. ⬠

The following basic lemma will be useful in the next section.

Lemma 3.6. *Let $K_1, K_2, \ldots$ be a sequence of compact subsets of $\mathbb{R}^2$ that converges to a Hausforff limit K. If the K_n are connected, then K is connected.*

4 Applications to Arnoux-Rauzy substitutions

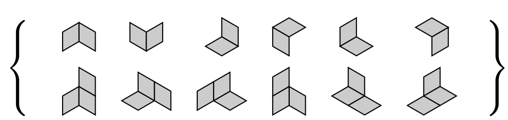
In this section, we use the tools developed in Section 3 to prove the connectedness of the images of $\mathcal{U} = [\mathbf{0}, 1]^* \cup [\mathbf{0}, 2]^* \cup [\mathbf{0}, 3]^*$ under any finite product of the generalized substitutions Σ_1, Σ_2, Σ_3 defined by

$$\Sigma_i = \mathbf{E}_1^*(\sigma_i) \qquad i \in \{1, 2, 3\}.$$

To this end, we introduce a finite set of connected patterns $\mathcal{L}_{\mathrm{AR}}$ (Equation 1) that covers $\mathcal{U}$, and that is stable under Σ_1, Σ_2 and Σ_3 (Proposition 4.1). The covering property will then be transferred to all the forward images of $\mathcal{U}$, which yields their connectedness (Theorem 4.2), and the connectedness of the fractal associated with a periodic product of Arnoux-Rauzy substitutions (Theorem 4.3).

Let

$$\mathcal{L}_{\mathrm{AR}} =$$

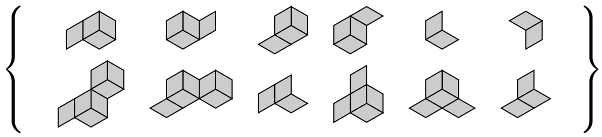


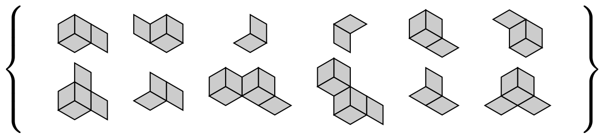
. (1)

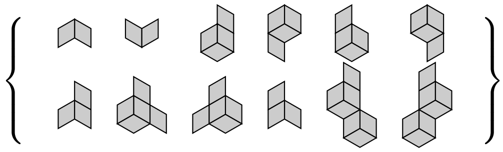
We do not give the explicit formula defining each pattern of $\mathcal{L}_{\mathrm{AR}}$, since it can clearly be deduced from the above graphical representation (we require that each pattern is a connected subset of $\mathbb{R}^3$). We also do not specify the position in $\mathbb{Z}^3$ of each pattern, because it doesn't matter: any choice is compatible with the proofs below.

Proposition 4.1. *The set of patterns $\mathcal{L}_{\mathrm{AR}}$ is stable under the generalized substitutions Σ_1, Σ_2 and Σ_3.*

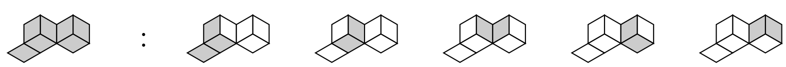
Proof. We must prove that every pattern of $\Sigma_i(\mathcal{L}_{\mathrm{AR}})$ is covered by $\mathcal{L}_{\mathrm{AR}}$, for $i = 1, 2, 3$, which makes a total of 36 patterns to check:

$\Sigma_1(\mathcal{L}_{\mathrm{AR}}) =$  ;

$\Sigma_2(\mathcal{L}_{\mathrm{AR}}) =$  ;

$\Sigma_3(\mathcal{L}_{\mathrm{AR}}) =$  .

This can easily be checked for each of these patterns, as for the following pattern of $\Sigma_1(\mathcal{L}_{\mathrm{AR}})$, for example:



From this graphical representation, we can deduce that there is an $\mathcal{L}_{\mathrm{AR}}$-chain between each two faces of the pattern. All the other patterns of $\Sigma_1(\mathcal{L}_{\mathrm{AR}})$ are also $\mathcal{L}_{\mathrm{AR}}$-covered, and the patterns of the sets $\Sigma_2(\mathcal{L}_{\mathrm{AR}})$ and $\Sigma_3(\mathcal{L}_{\mathrm{AR}})$ (which are symmetrical copies of the patterns of $\Sigma_1(\mathcal{L}_{\mathrm{AR}})$) admit similar $\mathcal{L}_{\mathrm{AR}}$-coverings. ⬠

We now use the stability of $\mathcal{L}_{\mathrm{AR}}$ and the results of Section 3 to obtain our connectedness results.

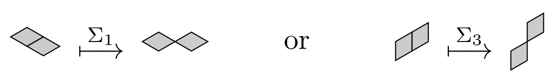
Theorem 4.2. *The set $\pi_{\mathrm{c}}(\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{U}))$ is connected, for every $i_1, \ldots, i_n \in \{1, 2, 3\}$.*

Proof. Proposition 3.4 implies that $\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{U})$ is $\mathcal{L}_{\mathrm{AR}}$-covered, because $\mathcal{L}_{\mathrm{AR}}$ is stable under the Σ_i (Proposition 4.1) and $\mathcal{U}$ is $\mathcal{L}_{\mathrm{AR}}$-covered. The projection by π_{c} of every pattern of $\mathcal{L}_{\mathrm{AR}}$ is connected, so Proposition 3.5 implies that the set $\pi_{\mathrm{c}}(\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{U}))$ is connected. ⬠

Since connectedness is preserved in the Hausdorff limit (Lemma 3.6), it follows that the fractals associated with Arnoux-Rauzy substitutions are connected.

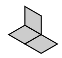
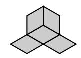
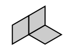
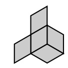
Theorem 4.3. *The Rauzy fractal associated with any Pisot irreducible finite product of Arnoux-Rauzy substitutions is connected.*

About the set $\mathcal{L}_{\mathrm{AR}}$ A first possible approach to obtain a stable set of patterns such as $\mathcal{L}_{\mathrm{AR}}$ is to consider the set of all patterns made of two faces sharing an edge. This strategy can be successful, as for example with substitutions associated with the Jacobi-Perron algorithm [IO93]. In the case of Arnoux-Rauzy substitutions, this set of patterns is not sufficient because there are some unavoidable patterns whose image is not edge-connected:



We could try to fix the problem by adding to our set all the two-face patterns that share a single vertex, but this is not enough because this new family is in turn not stable: we get some disconnected image patterns.

We have been able to find a stable set for Arnoux-Rauzy substitutions by "completing" the problematic patterns, *i.e.*, by considering some patterns of more than two faces:

 $\overset{\Sigma_1}{\longmapsto}$  or  $\overset{\Sigma_3}{\longmapsto}$  .

5 Further questions and counterexamples

In this section, we provide examples to show that some possible generalizations or extensions of Theorem 4.3 are not true.

The converse of Theorem 4.3 We give an example of a substitution σ whose Rauzy fractal is connected and such that no power of σ can be written as a product of Arnoux-Rauzy substitutions. This proves that the converse of Theorem 4.3 does not hold, *i.e.*, it is not true that every connected Rauzy fractal is associated with a product of Arnoux-Rauzy substitutions.

Let σ be the substitution defined by $1 \mapsto 32131$, $2 \mapsto 321$, $3 \mapsto 3213$. It is easy to check that there are at least 14 different words of length 6 in an infinite fixed point of σ. Hence, no power of σ is a product of Arnoux-Rauzy substitutions, because Arnoux-Rauzy sequences have complexity $2n + 1$. It can be checked algorithmically that the Rauzy fractal of σ is connected, using the methods described in [ST10] (see Figure 7a). Let us remark that these methods also enable us to prove that the Rauzy fractal of σ is also *simply* connected, which makes the counterexample even stronger.

Many other examples can easily be found, such as the substitution given in Figure 2a, whose Rauzy fractal is connected (but not simply connected).

Simple connectedness and Arnoux-Rauzy substitutions Simple connectedness of the Rauzy fractal does not hold in general for products of Arnoux-Rauzy substitutions. Indeed, the fractal associated with the substitution

$$\sigma = \sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\sigma_2\sigma_2\sigma_2\sigma_3\sigma_3\sigma_3\sigma_3$$

is not simply connected because σ is equal to the cube of the substitution given in Figure 2c, so it has the same Rauzy fractal, which has uncountable fundamental group [ST10].

Connectedness and invertible substitutions In the case of two-letter substitutions, it has been shown that the Rauzy fractal is connected if and only if the substitution is invertible [EI98]. (A substitution is *invertible* if it extends to an automorphism of the free group.)

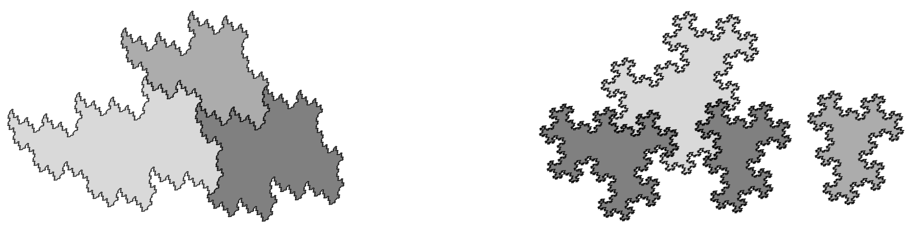
We can see Theorem 4.3 as a partial analogue of this fact for three-letter substitutions, because Arnoux-Rauzy substitutions are invertible. We are now going to see that Theorem 4.3 seems difficult to generalize to a larger class of invertible substitutions. Let

$$\varepsilon_{i,j} \;:\; \begin{cases} j \mapsto ij \\ k \mapsto k \text{ if } k \neq j \end{cases}$$

for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Every finite product of $\varepsilon_{i;j}$ is invertible and Arnoux-Rauzy substitutions can be written as products of $\varepsilon_{i;j}$: $\sigma_1 = \varepsilon_{1,2}\varepsilon_{1,3}$, $\sigma_2 = \varepsilon_{2,1}\varepsilon_{2,3}$, and $\sigma_3 = \varepsilon_{3,1}\varepsilon_{3,2}$. However the Rauzy fractal associated with the substitution

$$\varepsilon_{1,2}\varepsilon_{3,1}\varepsilon_{2,3}\varepsilon_{1,3} \;:\; \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 21 \\ 3 \mapsto 32113 \end{cases}$$

is not connected, as can be checked using the algorithms given in [ST10]. This fractal is depicted in Figure 7b. Let us mention that invertible substitutions on three letters have been characterized [TWZ04], and that finite products of $\varepsilon_{i;j}$ constitute only a proper subclass.



(a) $1 \mapsto 32131, 2 \mapsto 321, 3 \mapsto 3213$

(b) $1 \mapsto 13, 2 \mapsto 21, 3 \mapsto 32113$

Figure 7: (a) A simply connected Rauzy fractal where no power of σ can be written as a product of Arnoux-Rauzy substitutions. (b) A disconnected Rauzy fractal where σ is a product of elementary substitutions.

6 Conclusion

We have given a combinatorial proof of the connecteness of the Rauzy fractals associated with finite products of Arnoux-Rauzy substitutions. To do so, we have extended combinatorial techniques by considering coverings by patterns of more than two faces.

There is a lot of room for future work. We believe that the techniques introduced in this article will allow us to prove topological properties for some other families of Rauzy fractals. It would also be interesting to characterize the products of Arnoux-Rauzy substitutions which have simply connected Rauzy fractal.

There are also many interesting related decidability questions: given a unimodular Pisot irreducible subsitution, is its Rauzy fractal connected? Simply connected? Is the origin an

inner point? Does it verify the tiling property? Some of these questions have been addressed (see [ST10]), but the techniques used rely on incidence graphs of the subtiles of the fractals. It would be interesting to investigate and revisit these questions using the same techniques as in this article.

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